An integro-collocation method for determining initial values for ordinary differential equations

Lead Author

Adenipekun A. E.

Affiliation:

Department of Statistics, Federal Polytechnic, Ede, Osun State, Nigeria.

Abstract

The collocation method used to formulate an integrated Lanczos Tau method for solving ordinary differential equations with starting values is the subject of this research. In order to uniquely determine the coefficients of the approximant of the solution, an algebraic system of linear equations is created by collocating the perturbed integrated equation at certain equally spaced intervals within the range of integration of the differential equation. The method is used to solve problems involving first- and second-order ordinary differential equations, and data collected from numerical analysis supports its correctness and efficacy.

Keywords Lanczos Tau Method; Collocation method; Initial value problems; Chebyshev polynomial and ordinary differential equation.

Co-Authors

Adeniyi, R. B, Department of Mathematics, University of Ilorin, Ilorin, Kwara State, Nigeria. **Onanaye, A. S; Adeleke, O.J.**Department of Mathematics and Statistics, Redeemer's University, Ede, Osun State, Nigeria.

Introduction

Lanczos' and Ortiz's (1956 and 1969) Tau Method was originally based on numerical solutions of polynomial coefficient linear ordinary differential equations under certain supplementary (initial or boundary) conditions.

$$D\left[y(x)\right] = \sum_{i=0}^{m} P_i\left(x\right)\frac{d^i}{dx^i}y\left(x\right) = 0$$

$$D \cong \sum_{i=0}^{m} P_i\left(x\right)\frac{d^i}{dx^i} \tag{1.0.1}$$

By using Chebyshev polynomials, Lanczos (1956) introduced the concept of finite expansions when solving linear differential equations with polynomial coefficients

$$D\left[y(x)\right] = 0 \tag{1.0.2}$$

It has been proven that this method can be applied to a variety of scientific applications due to the fact that it is specially developed for providing economized representations of a variety of functions that are usually derivable from linear differential equations with polynomial coefficients, which are frequently used in scientific computation. It is a method for solving differential equations simply to approximate special functions of mathematical physics. Besides solving differential equations numerically, it comes in handy for solving integro-differential equations, stoke problems, and describing physical space-times in general. Equations with complex differentials and functional that can be solved numerically become easier with the help of it. An approximate solution is obtained by solving exactly an approximate problem which is the main goal in it.

Instead of truncating an infinite expansion of power series in an effort to find an nth order approximate solution of (1.0.2), A modified version of Lanczos' procedure is searched for an exact polynomial solution, which is the perturbed equation of (1.0.1) that is modified by adding a polynomial perturbation term to the right hand side. Power series solutions can be found with only a finite number of terms other than zero if the term is chosen in such a way.

Mohammad and M. Mohammad (2022) introduced the idea of the Chebyshev Tau method in order to linear Klein Gordon equation with the use Maple software to digitally solve the problem.

Introduction to tau's integrated formula

It was often an integrated approach that led to an improvements in the accuracy of the Tau approximant y(x) gotten from a differential form. As a result of this variant, the differential equation is integrated through an order of time as many times as its order, resulting in a higher order perturbed differential equation.

The Method and Derivation

Suppose we have a linear differential system of m-th order;

$$Ly(x) \equiv P_0(x)y(x) + P_1(x)y^{/}(x) + P_2(x)y^{//}(x) + \ldots + P_m(x)y^{m}(x) = f(x) \qquad (2.1.1)$$

where $a \le x \le b$

$$L^{*}y(x_{rk}) = \sum_{r=0}^{m} a_{rk}\, y^{r}(x_{rk}) = \ell_k, \qquad k = 1(1)m \qquad (2.1.2)$$

Let $\int \int \int \ldots^{m} \int g(x)\, dx$ denotes the indefinite integration m times applied to the function $g(x)$ and

$$I_L = \int \int \int \ldots \int L(.)\, dx \qquad (2.1.3)$$

The integrated form of (2.1.1) is written as

$$I_L = \int \int \int \ldots^{m} \int f(x)\, dx + c_m(x) \qquad (2.1.4)$$

Where $c_m(x)$ denotes an arbitrary polynomial of degree (m - 1) arising from the constants of the integration. The Tau approximant $y_n(x)$ thus satisfies the perturbed problem;

$$I_L = \int \int \int \ldots^{m} \int f(x)\, dx + c_m(x) + H_{m+n}(x) \qquad (2.1.5)$$

where

$$H_{m+n}(x) = \sum_{r=0}^{m+s+1} \tau_{m+s-1} T_{n-m+r+1}(x) \tag{2.1.6}$$

The Tau method often gives a more accurate approximation due to the higher order perturbation term. To combine this variant of the Tau method with collocation, we collocate (2.1.5) – (2.1.6) at the equally spaced points.

$$x_k = a + \frac{(b-a)k}{n+2}, \qquad k = 1(1)n+2, \ \forall\ x_k \in [a, b] \tag{2.1.7}$$

This leads to a system of (n + 2) algebraic equations for uniquely determining the (n + 1) coefficients of

$$y_n(x) = \sum_{r=0}^{n} a_r\, x_r \cong y(x)$$

(2.1.8)

and the Tau (τ) parameter.

This is the exact solution of a perturbed equation by adding a polynomial perturbation term to the right hand side of (2.1.1). The polynomial $y_n(x)$ satisfies the differential system

$$L y_n(x) = \sum_{r=0}^{m} P_r(x) y_n^{(r)}(x) = f(x) + H_n(x) \tag{2.1.9}$$

$$L^* y_n(x) = \sum_{r=0}^{m-1} a_{rk}(x) y_n^{(r)}(x_{rk}) = \alpha_k, \qquad k = 1(1)m \tag{2.1.10}$$

where the perturbation term is constructed in such a way that (2.1.9) and (2.1.10) has a polynomial solution of degree n. Oyedotun, et. al. (2021) introduced error analysis of the Tau method in order to a class of third order initial value problem.

Lanczos (1956) took $H_n(x)$ to be a linear combination of powers of x multiplied by the Chebyshev polynomials. The choice of the Chebyshev polynomials stems from the desire to distribute the errors defined by;

$$\max_{a \le x \le b} \left| y(x) - y_n(x) \right|$$

CHEBYSHEV POLYNOMIAL

The Chebyshev polynomial, $T_n(x)$, of degree n over the interval [-1, 1] is defined by

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, 2, \ldots \tag{3.0.1}$$

This

expression can also be written as

$$T_n(x) = \cos n\theta, \quad where \quad x = \cos\theta \tag{3.0.2}$$

From equation (3.1), it is easy to observe that

$T_n(x) = T_{-n}(x)$. Also, $T_{2n}(x) = T_{2n}(-x)$ and $T_{2n+1}(-x) = -T_{2n+1}(x)$,

i.e

$$T_n(x) \tag{3.0.3}$$

is even or odd function according as n is even or odd. Now, from the trigonometric formula;

$$\cos(n-1)\theta + \cos(n+1)\theta = 2\cos n\theta \cos\theta \tag{3.0.4}$$

The recurrence relation for Chebyshev polynomial is obtained as

$T_{n-1}(x) + T_{n+1}(x) = 2xT_n(x)$

which can be written as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, 3, \ldots \tag{3.0.5}$$

From (3.0.2), it is obvious that $\left| T_n(x) \right| \le 1$ $for -1 \le x \le 1$. We can generate the Chebyshev polynomial from (3.0.2) and (3.0.4);

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Shifted Chebyshev Polynomials

The interval of the Chebyshev polynomial $-1 \le x \le 1$ will be converted to the interval $0 \le x \le 1$ because of its usefulness in the methodology. This conversion of the interval of Chebyshev polynomial to another is what we call the "Shifted Chebyshev Polynomial".

If $X \in [-1, 1] \rightarrow U \in [0, 1]$ and let $U = \alpha x + \beta,$ where α and β are some parameters. Therefore,
$-\alpha + \beta = 0$ *and* $\alpha + \beta = 1$, solving the equations simultaneously we have that $\alpha = \beta = \frac{1}{2}$. Substitute back the values of α and β to have $X = 2U - 1$. Thus, we have the shifted Chebyshev polynomial
$T_n^*(x) = T_n^*(2x - 1)$, (since U is a dummy variable)

Hence;

$T_0(x) = 1 \Rightarrow T_n^*(x) = 1$

$T_1(x) = x \Rightarrow T_1^*(x) = 2x - 1$

$T_2(x) = 2x^2 - 1 \Rightarrow T_2^*(x) = 8x^2 - 8x + 1$

$T_3(x) = 4x^3 - 3x \Rightarrow T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$

$T_4(x) = 8x^4 - 8x^2 + 1 \Rightarrow T_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x \Rightarrow T_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \Rightarrow T_6^*(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$

The collocation method

A collocation method is a method which involves finding an approximation solution to an equation based on a set of functions, sometimes referred to as the trial function or basis function, where the approximate solution has to satisfy certain conditions at certain points of the domain of definition, called the Collocation points.

The method was first proposed by Kantorovich and Akilov (2009). Their method was a procedure for collocating lines in two variables for the solution of partial differential equations, in which the collocation was applied to one variable for every fixed value in the second variable.
Within the specified range of the problem, standard collocation requires equal spacing between collocation points, i.e.

$$x_k \in [a, b], \quad x_k = kh, \quad k = 1(1)(k+1), \qquad where\ \ h = \frac{b-a}{n+1}$$

DESCRIPTION OF THE METHOD

In the description of the method, we shall consider the first and second order differential equations

Numerical Example on First Order Differential Equation

Let us consider this;

$$2(1+x)\frac{dy}{dx} + y = 0, \qquad at\ \ y(0) = 1$$

Whose theoretical solution is

$$y = \frac{1}{\sqrt{1+x}}$$

To apply the technique, we will integrate once since it is of first order differential equation

$$2(1+x)\frac{dy}{dx}+y=0 \quad \Rightarrow \quad 2y^{/}(x)+2xy^{/}(x)+y(x)=0$$

$$I_L\big(y(x)\big)=2\int_0^x y^{/}(s)\,ds+2\int_0^x s\,y^{/}(s)\,ds+\int_0^x y(s)\,ds=\tau_1 T_{n+1}(x)$$

$$2[y(x)-y(0)]+2[xy(x)-\int_0^x y(s)\,ds]+\int_0^x y(s)\,ds=\tau_1 T_{n+1}(x)$$

$$2[y(x)-y(0)]+2xy(x)-2\int_0^x y(s)\,ds+\int_0^x y(s)\,ds=\tau_1 T_{n+1}(x)$$

$$2[y(x)-1]+2xy(x)-\int_0^x y(s)\,ds=\tau_1 T_{n+1}(x) \qquad \text{since } y(0)=1$$

$$2y(x)-2+2xy(x)-\int_0^x y(s)\,ds=\tau_1 T_{n+1}(x)$$

But $y_n(x)=\sum_{r=0}^{n} a_r x^r$, then we have

$$2\sum_{r=0}^{n} a_r x^r-2+2x\sum_{r=0}^{n} a_r x^r-\sum_{r=0}^{n}\frac{a_r x^{r+1}}{r+1}=\tau_1 T_{n+1}(x)$$

$$2\sum_{r=0}^{n} a_r x^r+2\sum_{r=0}^{n} a_r x^{r+1}-\sum_{r=0}^{n}\frac{a_r x^{r+1}}{r+1}=2+\tau_1 T_{n+1}(x)$$

Now, we take the degree of Y -approximant at $N=3$

$$2\sum_{r=0}^{3} a_r x^r + 2\sum_{r=0}^{3} a_r x^{r+1} - \sum_{r=0}^{3} \frac{a_r x^{r+1}}{r+1} = 2 + \tau_1 T_4(x)$$

$$2\sum_{r=0}^{3} a_r x^r + 2\sum_{r=0}^{3} a_r x^{r+1} - \sum_{r=0}^{3} \frac{a_r x^{r+1}}{r+1} = 2 + \tau_1 \sum_{r=0}^{4} C_r^{n+1} x^r$$

$$2\sum_{r=0}^{3} a_r x^r + 2\sum_{r=0}^{3} a_r x^{r+1} - \sum_{r=0}^{3} \frac{a_r x^{r+1}}{r+1} = 2 + \tau_1 \sum_{r=0}^{4} C_r^{4} x^r$$

$$2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) + 2\left(a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4\right) - \left(a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3 + \frac{1}{4}a_3 x^4\right) =$$
$$2 + \tau_1\left(128x^4 - 256x^3 + 160x^2 - 32x + 1\right)$$

We collocate at $x_k = kh$, where $h = \frac{1}{n+2} \Rightarrow h = \frac{1}{5}$. Therefore,

$x_k = \frac{k}{5}, \quad k = 1(1)5$

Which implies that $x_1 = \frac{1}{5},\ x_2 = \frac{2}{5},\ x_3 = \frac{3}{5},\ x_4 = \frac{4}{5},\ x_5 = 1$

At $x = \frac{1}{5}$,

$$2\left(a_0 + \frac{1}{5}a_1 + \frac{1}{25}a_2 + \frac{1}{125}a_3\right) + 2\left(\frac{1}{5}a_0 + \frac{1}{25}a_1 + \frac{1}{125}a_2 + \frac{1}{625}a_3\right) - \left(\frac{1}{5}a_0 + \frac{1}{50}a_1 + \frac{1}{375}a_2 + \frac{1}{2500}a_3\right) =$$

$$2 + \tau_1\left(\frac{128}{625} - \frac{256}{125} + \frac{160}{25} - \frac{32}{5} + 1\right)$$

$$\Rightarrow\ 16500a_0 + 3450a_1 + 700a_2 + 141a_3 + 6324\tau_1 = 15000 \qquad (1)$$

At $x = \frac{2}{5}$,

$$2\left(a_0 + \frac{2}{5}a_1 + \frac{4}{25}a_2 + \frac{8}{125}a_3\right) + 2\left(\frac{2}{5}a_0 + \frac{4}{25}a_1 + \frac{8}{125}a_2 + \frac{16}{625}a_3\right) - \left(\frac{2}{5}a_0 + \frac{2}{25}a_1 + \frac{8}{375}a_2 + \frac{4}{625}a_3\right) =$$

$$2 + \tau_1\left(128\left(\frac{16}{625}\right) - 256\left(\frac{8}{125}\right) + 160\left(\frac{4}{25}\right) - 32\left(\frac{2}{5}\right) + 1\right)$$

$$\Rightarrow\ 4500a_0 + 1950a_1 + 800a_2 + 324a_3 - 1299\tau_1 = 3750 \qquad (2)$$

At $x = \frac{3}{5}$,

$$2\left(a_0 + \frac{3}{5}a_1 + \frac{9}{25}a_2 + \frac{27}{125}a_3\right) + 2\left(\frac{3}{5}a_0 + \frac{9}{25}a_1 + \frac{27}{125}a_2 + \frac{81}{625}a_3\right) - \left(\frac{3}{5}a_0 + \frac{9}{50}a_1 + \frac{27}{375}a_2 + \frac{81}{625}a_3\right) =$$

$$2 + \tau_1\left(128\left(\frac{81}{625}\right) - 256\left(\frac{27}{125}\right) + 160\left(\frac{9}{25}\right) - 32\left(\frac{3}{5}\right) + 1\right)$$

$$\Rightarrow\ 19500a_0 + 13050a_1 + 8100a_2 + 4941a_3 - 5196\tau_1 = 15000 \qquad (3)$$

At $x = \frac{4}{5}$,

$$2\left(a_0 + \frac{4}{5}a_1 + \frac{16}{25}a_2 + \frac{64}{125}a_3\right) + 2\left(\frac{4}{5}a_0 + \frac{16}{25}a_1 + \frac{64}{125}a_2 + \frac{256}{625}a_3\right) - \left(\frac{4}{5}a_0 + \frac{8}{25}a_1 + \frac{64}{375}a_2 + \frac{64}{625}a_3\right) =$$

$$2 + \tau_1\left(128\left(\frac{256}{625}\right) - 256\left(\frac{64}{125}\right) + 160\left(\frac{16}{25}\right) - 32\left(\frac{4}{5}\right) + 1\right)$$

$$\Rightarrow\ 5250a_0 + 4800a_1 + 4000a_2 + 3264a_3 + 1581\tau_1 = 3750 \qquad (4)$$

At $x = 1,$

$$2\left(a_0 + a_1 + a_2 + a_3\right) + 2\left(a_0 + a_1 + a_2 + a_3\right) - \left(a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3\right) = 2 + \tau_1\left(128 - 256 + 160 - 32 + 1\right)$$

$$\Rightarrow 36a_0 + 42a_1 + 44a_2 + 45a_3 - 12\tau_1 = 24 \qquad (5)$$

Solving (1), (2), (3), (4) and (5) simultaneously, we have;

$a_0 = 0.999437880434$, $a_1 = -0.481731114121$, $a_2 = 0.271359205109$, $a_3 = -0.0822300621$

$\tau_1 = -0.001124239131$

For the $Y-$approximant at $n = 3$;

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \Rightarrow y_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Now, substitute the values of a_0, a_1, a_2, a_3 to have the $Y-$approximant

$$y_3(x) = 0.9994378804 \quad 34 - 0.4817311141 \quad 21\ x + 0.2713592051 \quad 09\ x^2 - 0.0822300621 \quad 54\ x^3$$

The table 1 shows the exact error evaluated at some equidistance points.

x	$Y-Approximant$	$Analytical\ solution$	$Exact\ Error$
0.1	0.953896131011	0.953462589246	$4.33541765 \times 10^{-4}$
0.2	0.913863795752	0.912870929175	9.928666×10^{-4}
0.3	0.877120662979	0.877058019307	6.26436×10^{-5}
0.4	0.844900183625	0.845154254729	2.540711×10^{-4}
0.5	0.816133366882	0.816496580928	3.632141×10^{-4}
0.6	0.790326832375	0.790569415042	2.425827×10^{-4}
0.7	0.766987199734	0.766964988847	2.22109×10^{-5}
0.8	0.745621088584	0.7453559925	2.65096×10^{-4}
0.9	0.725735118553	0.72547625011	2.588684×10^{-4}

1.0	0.706835909265	0.707106781187	2.708719×10^{-4}

Numerical Example on Second Order Differential Equation

Let us consider this;

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x, \qquad at \quad x = 0,\ y = 0,\ \frac{dy}{dx} = -1$$

Whose theoretical solution is

$$y = 2e^{-x} + x - 2$$

To apply the technique, we will integrate twice since it is of second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x \quad \Rightarrow \quad y''(x) + 2y'(x) + y(x) = x$$

$$I_L\left(y(x)\right) = \int_0^x \int_0^v \left[y''(u) + 2y'(u) + y(u) \right] du\, dv = \left[\int_0^x \int_0^v u\, du\, dv \right] + \tau_1 T_{n+1}(x)$$

$$\int_0^x \left[\int_0^v y''(u) + 2y'(u) + y(u)\, du \right] dv = \int_0^x \left[\int_0^v u\, du \right] dv + \tau_1 T_{n+1}(x)$$

$$\int_0^x \left(\left[y'(u) + 2y(u) \right]_0^v + \int_0^v y(u)\, du \right) dv = \int_0^x \left[\frac{u^2}{2} \right]_0^v dv + \tau_1 T_{n+1}(x)$$

$$\int_0^x \left[\left(y'(v) + 2y(v) - y'(0) - 2y(0) \right) + \sum_{r=0}^{n} \frac{a_r v^{r+1}}{r+1} \right] dv = \frac{1}{2} \int_0^x v^2\, dv + \tau_1 T_{n+1}(x)$$

Substituting the value of the conditions given, we have

$$\int_0^x \left[\left(y^{/}(v) + 2y(v) + 1 \right) + \sum_{r=0}^{n} \frac{a_r v^{r+1}}{r+1} \right] dv = \frac{1}{2} \int_0^x v^2 \, dv + \tau_1 T_{n+1}(x)$$

$$\left[y(v) + v \right]_0^x + 2\int_0^x y(v) dv + \int_0^x \sum_{r=0}^{n} \frac{a_r v^{r+1}}{r+1} dv = \frac{1}{6} \left[v^3 \right]_0^x + \tau_1 T_{n+1}(x)$$

$$y(x) + x - y(0) + 2 \sum_{r=0}^{n} \frac{a_r x^{r+1}}{r+1} + \sum_{r=0}^{n} \frac{a_r x^{r+2}}{x+2} = \frac{1}{6} x^3 + \tau_1 T_{n+1}(x)$$

$$y(x) + 2 \sum_{r=0}^{n} \frac{a_r x^{r+1}}{r+1} + \sum_{r=0}^{n} \frac{a_r x^{r+2}}{x+2} = \frac{1}{6} x^3 - x + \tau_1 T_{n+1}(x)$$

Now, we take the degree of approximation at $N = 3$,

$$\sum_{r=0}^{3} a_r x^r + 2 \sum_{r=0}^{3} \frac{a_r x^{r+1}}{x+1} + \sum_{r=0}^{3} \frac{a_r x^{r+2}}{r+2} = \frac{1}{6} x^3 - x + \tau_1 \sum_{r=0}^{4} C_r^4 x^r$$

$$\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 \right) + 2\left(a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \frac{1}{4} a_3 x^4 \right) + \left(\frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{4} a_2 x^4 + \frac{1}{5} a_3 x^5 \right) =$$

$$\frac{1}{6} x^3 - x + \tau_1 \left(128 x^4 - 256 x^3 + 160 x^2 - 32 x + 1 \right)$$

We collocate at $x_k = kh$, where $h = \dfrac{1}{n+2} \Rightarrow h = \dfrac{1}{5}$. Therefore,

$$x_k = \frac{k}{5}, \quad k = 1(1)5$$

This implies that $x_1 = \frac{1}{5},\ x_2 = \frac{2}{5},\ x_3 = \frac{3}{5},\ x_4 = \frac{4}{5},\ x_5 = 1$

At $x = \frac{1}{5}$,

$$\left(a_0 + \frac{1}{5}a_1 + \frac{1}{25}a_2 + \frac{1}{125}a_3\right) + 2\left(\frac{1}{5}a_0 + \frac{1}{50}a_1 + \frac{1}{375}a_2 + \frac{1}{2500}a_3\right) + \left(\frac{1}{50}a_0 + \frac{1}{375}a_1 + \frac{1}{2500}a_2 + \frac{1}{15625}a_3\right) =$$

$$\frac{1}{750} - \frac{1}{5} + \tau_1\left(\frac{128}{625} - \frac{256}{125} + \frac{160}{25} - \frac{32}{5} + 1\right)$$

$$\Rightarrow 1331250a_0 + 227500a_1 + 42875a_2 + 8310a_3 + 790500\tau_1 = -186250 \qquad (1)$$

At $x = \frac{2}{5}$,

$$\left(a_0 + \frac{2}{5}a_1 + \frac{4}{25}a_2 + \frac{8}{125}a_3\right) + 2\left(\frac{2}{5}a_0 + \frac{4}{50}a_1 + \frac{8}{375}a_2 + \frac{16}{2500}a_3\right) + \left(\frac{4}{50}a_0 + \frac{8}{375}a_1 + \frac{16}{2500}a_2 + \frac{32}{15625}a_3\right) =$$

$$\frac{8}{750} - \frac{2}{5} + \tau_1\left(128\left(\frac{16}{625}\right) - 256\left(\frac{8}{125}\right) + 160\left(\frac{4}{25}\right) - 32\left(\frac{2}{5}\right) + 1\right)$$

$$\Rightarrow 1762500a_0 + 545000a_1 + 196000a_2 + 73920a_3 - 649500\tau_1 = -360000 \qquad (2)$$

At $x = \frac{3}{5}$,

$$\left(a_0 + \frac{3}{5}a_1 + \frac{9}{25}a_2 + \frac{27}{125}a_3\right) + 2\left(\frac{3}{5}a_0 + \frac{9}{50}a_1 + \frac{27}{375}a_2 + \frac{81}{2500}a_3\right) + \left(\frac{9}{50}a_0 + \frac{27}{375}a_1 + \frac{81}{2500}a_2 + \frac{243}{15625}a_3\right) =$$

$$\frac{27}{750} - \frac{3}{5} + \tau_1\left(128\left(\frac{81}{625}\right) - 256\left(\frac{27}{125}\right) + 160\left(\frac{9}{25}\right) - 32\left(\frac{3}{5}\right) + 1\right)$$

$$\Rightarrow 2231250a_0 + 967500a_1 + 502875a_2 + 277830a_3 - 649500\tau_1 = -528750 \qquad (3)$$

At $x = \frac{4}{5}$,

$$\left(a_0 + \frac{4}{5}a_1 + \frac{16}{25}a_2 + \frac{64}{125}a_3\right) + 2\left(\frac{4}{5}a_0 + \frac{16}{50}a_1 + \frac{64}{375}a_2 + \frac{256}{2500}a_3\right) + \left(\frac{16}{50}a_0 + \frac{64}{375}a_1 + \frac{256}{2500}a_2 + \frac{1024}{15625}a_3\right) =$$

$$\frac{64}{750} - \frac{4}{5} + \tau_1\left(128\left(\frac{256}{625}\right) - 256\left(\frac{64}{125}\right) + 160\left(\frac{16}{25}\right) - 32\left(\frac{4}{5}\right) + 1\right)$$

$$\Rightarrow 2737500a_0 + 1510000a_1 + 1016000a_2 + 733440a_3 + 790500\tau_1 = -670000 \qquad (4)$$

At $x = 1$,

$$\left(a_0 + a_1 + a_2 + a_3\right) + 2\left(a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3\right) + \left(\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3\right) = \frac{1}{6} - 1 + \tau_1\left(128 - 256 + 160 - 32 + 1\right)$$

$$\Rightarrow 105a_0 + 70a_1 + 57.5a_2 + 51a_3 - 30\tau_1 = -25 \qquad (5)$$

Having solved (1), (2), (3), (4) and (5) simultaneously, we have;

$a_0 = 0.001793702910$, $a_1 = -1.015646447$, $a_2 = 1.052558639$, $a_3 = -0.2868133318$
$\tau_1 = -0.0004090234788$

For the $Y-$approximant at $n = 3$;

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \Rightarrow y_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Now, substitute the values of a_0, a_1, a_2, a_3 to have the $Y-$approximant

$$y_3(x) = 0.0017937029\,10 - 1.015646447\,x + 1.052558639\,x^2 - 0.2868133318\,x^3$$

The table 2 shows the exact error evaluated at some equidistance points.

x	$Y-Approximant$	$Analytical\ solution$	$Exact\ Error$
0.1	$-0.0895321687\,32$	$-0.0090325163\,804$	7.9299507×10^{-4}
0.2	$-0.1625384936\,29$	$-0.1625384936\,29$	1.0107461×10^{-3}
0.3	$-0.2159139136\,39$	$-0.2183635591\,02$	2.4496455×10^{-3}
0.4	$-0.2544115468\,85$	$-0.2593599082\,51$	4.9483614×10^{-3}
0.5	$-0.2787415273\,15$	$-0.2869386807\,95$	8.1971534×10^{-3}
0.6	$-0.2906247349\,19$	$-0.3023767275\,77$	1.1751992×10^{-2}
0.7	$-0.2917820496\,87$	$-0.3068293921\,85$	1.5047342×10^{-2}
0.8	$-0.2839343516\,12$	$-0.3013420715\,67$	1.7407719×10^{-2}
0.9	$-0.2688025206\,82$	$-0.2868606804\,67$	1.8058159×10^{-2}
1.0	$-0.2481074368\,9$	$-0.2642411175\,33$	1.613368×10^{-2}

CONCLUSION

A numerical method which combines the idea of the integrated formulation of the Tau method with collocation techniques has been presented. The integration of the differential equation ensures that the corresponding Tau formulation involves a perturbation term of higher order to guarantee an improved accuracy. The collocation also allows for the choice of sufficient number of points at which the resulting integrated equation is evaluated to guarantee a consistent linear system of algebraic equations to make for solvability. First and second order differential equations have been considered for the illustration of the scheme developed and the numerical evidences confirmed that the technique is accurate and effective.

Also, it could be observed that the higher the order of the differential equation, the less the accuracy of the technique. This was confirmed from the tables that the higher the order of the degree the less the accuracy the method could give.

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